NONTANGENTIAL CONVERGENCE OF THE GENERALIZED POISSON-ABEL MEANS

Nakhman Alexander D.
Tambov State Technical University, Tambov, Russia
e-mail: alextmb@mail.ru

Abstract. The means of Fourier series \( U(f, y; \lambda, h) \) generated by semi-continuous summation methods \( \Lambda = \{ \lambda_k(h), k = 0, 1, \ldots; h > 0 \} \) are studied. For the points \( (y, h) \), belonging to an angular domain \( \Gamma_d(x) \), upper \( L^p \)-estimates of the corresponding maximal operators are obtained. Nontangential convergence almost everywhere of the generalized Poisson-Abel means, corresponding to a case of \( \lambda_k(h) = \exp(-hk^\alpha), \ k = 0, 1, \ldots; \ \alpha \geq 1 \), is established.

Keywords. Exponential summation methods, estimates of \( L^p \)-norms, nontangential convergence.

1. Introduction. Formulation of the problem. Let \( L_{2\pi} \) be class of \( 2\pi \)-periodical functions, which are summable on \([-\pi, \pi]\) and \( C^2(0, +\infty) \) – class of functions having continuous second derivative on \((0, +\infty)\). In this paper we consider the semi-continuous means

\[
U(f, y; \lambda, h) = \sum_{k=-\infty}^{\infty} \lambda_k(h) c_k(f) \exp(iky)
\]

of Fourier series \( s[f] \) of functions \( f \in L_{2\pi} \). In the definition (1)

\[
c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp(-ikt) dt, \ k = 0, \pm 1, \pm 2, \ldots
\]

are complex Fourier coefficients of function \( f \).

We study the problem of behavior (1) at \( (y, h) \to (x, 0) \), when the point \( (y, h) \) is within the boundaries of the angular domain

\[
\Gamma_d(x) = \{(y, h) \mid y \in [-\pi, \pi], h > 0, \ |y-x|_h \leq d\}, \ d = const, \ d > 0.
\]

The case of “radial” convergence \( U(f, x; \lambda, h) \to f(x) \) at \( h \to +0 \) was investigated in [1].

2. The main result. Define

\[
U_*(f) = U_*(f, x; \lambda) = \sup_{(y, h) \in \Gamma_d(x)} |U(f, y; \lambda, h)|;
\]

let \( m = \left\lfloor \frac{1}{2dh} \right\rfloor \) and

\[
f^*(x) = \sup_{\eta>0} \frac{1}{2\eta} \int_{x-\eta}^{x+\eta} |f(t)| \, dt
\]

be Hardy maximal function ([2], vol.1, p.55).

**Theorem 1.** Let the sequence \( \{\lambda_N(h)\} \) decreases so rapidly that

\[
N |\lambda_N(h)| + N^2 |\Delta \lambda_N(h)| = o(1), \ N \to \infty, \tag{2}
\]

and there is a constant \( C = C_{\lambda,d} \) such that
\[
\sum_{k=0}^{\infty} \frac{(m+k+1)(k+1)}{m} |\Delta^2 \lambda_k(h)| \leq C.
\]  
(3)

Then for every \( x \) the estimate

\[
U^*(f, x; \lambda) \leq C_{\Lambda, d} \; f^*(x)
\]

holds.

Here and throughout the paper \( C \) will represent constants, which depend only on the explicitly specified indexes.

3. \( L^p \)-estimates. Let

\[
\| f \|_p = \left( \int_{-\pi}^{\pi} |f(x)|^p \, dx \right)^{1/p}
\]

be a norm in Lebesgue space \( L^p \) (\( p > 0 \); \( L = L^1 \); \( \| f \| = \| f \|_1 \)).

**Theorem 2.** If the sequence \( \Lambda \) satisfies the conditions (2) and (3), the following estimates

\[
\| U^*(f) \|_p \leq C_{p, \Lambda} \| (f) \|_p, \; p > 1;
\]

\[
\| U^*(f) \| \leq C_{\Lambda} (1+ \| (f) \|) ;
\]

\[
\| U^*(f) \|_p \leq C_{p, \Lambda} \| (f) \|, \; 0 < p < 1 .
\]

(4)

hold.


**Theorem 3.** If \( f \in L_{2\pi} \), the sequence \( \Lambda \) satisfies (2), (3) and

\[
\lim_{h \to 0} \lambda_k(h) = 1, \; k = 0, 1, \ldots,
\]

then the relation

\[
\lim_{(y,h)\to(x,0)} U(f, y; \lambda, h) = f(x)
\]

holds almost everywhere.

This theorem can be proved by the standard method ([1], vol. 2, pp. 464-465) due to the estimate (4).

4. Exponential means. Denote now

\[
\lambda_0(h) = 1, \quad \lambda_k(h) = \lambda(x, h) |_{x=k}, \; k = 1, 2, \ldots,
\]

where \( \lambda(x, h) = \exp(-h \varphi(x)) \), and require the following conditions:

A) \( \varphi \in C^2(0, +\infty); \; \varphi(x) \geq 0, \; \varphi'(x) \geq 0, \; \varphi''(x) \geq 0, \; x \in (0, +\infty) ;
\)

B) \( x^2 \left( \varphi'(x) \right)^2 \exp(-h \varphi(x)) \) and \( x^2 |\varphi''(x)| \exp(-h \varphi(x)) \) decrease to zero as \( x \) increases.

Note that

\[
\lambda''_x(x,h) = h \exp(-h \varphi(x))(h \varphi'(x))^2 - \varphi''(x)
\]

and apply twice the Lagrange theorem to the second finite differences in (3).

Under the conditions of B) the sum of (3) is majorized by a corresponding improper integral and for implementability of statements of Theorems 1, 2, 3 it is sufficient to require

\[
\int_{0}^{\infty} (h^2 \varphi'(x))^2 + h |\varphi''(x)| (x + h x^2) \exp(-h \varphi(x)) \, dx \leq C_{\varphi}.
\]

5. Generalized Poisson-Abel means. Consider in particular the case \( \varphi(x) = x^{\alpha}, \; \alpha \geq 1 \), then

\[
\lambda_0(h) = 1, \; \lambda_k(h) = \exp(-hk^{\alpha}), \; k = 1, 2, \ldots; \; \alpha \geq 1.
\]

**Corollary 1.** The statements of Theorems 2 and 3 are valid for generalized Poisson-Abel means.
\[ \sigma(f, y; \alpha, h) = \sum_{k=-\infty}^{\infty} \exp(-h |k|^\alpha) c_k(f) \exp(iky) \]

for all \( \alpha \geq 1 \); the constants \( C \) in the estimates of \( L^p \)-norms is \( C = C_{\alpha, p} \).

In particular, the relation

\[ \lim_{(y, h) \to (x, 0)} \sum_{k=-\infty}^{\infty} \exp(-h |k|) c_k(f) \exp(iky) = f(x), \ f \in L_{2\pi}, \]

(nontangential convergence of Poisson-Abel means) holds for almost all \( x \).

6. Exponentially-polynomial summation methods. Let now \( \phi(x) \) is a polynomial function of \( n \)-th degree

\[ P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \ a_n > 0; \ n = 1,2,\ldots \]

**Corollary 2.** The assertions of Theorems 2 and 3 are valid for exponentially-polynomial means

\[ \sigma(f, y; n, h) = \sum_{k=-\infty}^{\infty} \exp(-h P_n(|k|)) c_k(f) \exp(iky) \]

for all \( n = 1,2,\ldots \); the constants \( C \) in the estimates of \( L^p \)-norms is \( C = C_{n, p} \).

**References**